QUANTUM COHOMOLOGY AND THE PERIODIC TODA LATTICE

Martin A. Guest and Takashi Otofuji

Abstract. We describe a relation between the periodic one-dimensional Toda lattice and the quantum cohomology of the periodic flag manifold (an infinite-dimensional Kähler manifold). This generalizes a result of Givental and Kim relating the open Toda lattice and the quantum cohomology of the finite-dimensional flag manifold. We derive a simple and explicit "differential operator formula" for the necessary quantum products, which applies both to the finite-dimensional and to the infinite-dimensional situations.

Introduction.

The quantum cohomology of the full flag manifold F_n of SU_n is known to be related to an integrable system, the open one-dimensional Toda lattice. This relation was established in [Gi-Ki], and a rigorous framework for the calculations was developed in [Ci1], [Ki1], and [Lu], building on earlier fundamental work in quantum cohomology. We shall give — in the spirit of [Gi-Ki] — an analogous relation between the quantum cohomology of the periodic flag manifold $Fl^{(n)}$ and the periodic one-dimensional Toda lattice.

Such an extension to the periodic case is perhaps not unexpected, but we feel that it is worth noting, for two reasons. First, there are several new features of the quantum cohomology of the periodic flag manifold $Fl^{(n)}$, the most obvious one being that $Fl^{(n)}$ is an *infinite-dimensional* Kähler manifold. Second, very few concrete examples of this phenomenon are known (cf. section 2.3 of [Au]). Indeed, the full flag manifold F_n seems to be the only example so far, together with its generalization[1] G/B which was accomplished in [Ki2]. Now, $Fl^{(n)}$ is an infinite-dimensional flag manifold (of the loop group LSU_n), and is therefore a close relative of this family. However, the periodic one-dimensional Toda lattice is more complicated than the open one; for example its solutions generally involve theta functions, whereas those of the open Toda lattice are rational expressions of exponential functions.

[1] Some comments on the case of partial flag manifolds and their relation with Toda lattices can be found in section 5 of [Gi1].

The open one-dimensional Toda lattice is a (nonlinear) first-order differential equation

$$\dot{L}_n(t) = [L_n(t), M_n(t)]$$

where L_n is the tri-diagonal matrix

$$L_n = \begin{pmatrix} X_1 & Q_1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & X_2 & Q_2 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & X_3 & Q_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & X_{n-2} & Q_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & -1 & X_{n-1} & Q_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & X_n \end{pmatrix}$$

and M_n is a certain modification of L_n. Here, $X_1, \ldots, X_n$ and $Q_1, \ldots, Q_{n-1}$ are functions of a real variable t with $Q_i < 0$, and we assume that $X_1 + \cdots + X_n = 0$. Let

$$\det(L_n + \mu I) = O_n = \sum_{i=0}^{n} O_n^i \mu^i.$$

Then the polynomials $O_n^0, O_n^1, \ldots, O_n^{n-1}$ in $X_1, \ldots, X_n$ and $Q_1, \ldots, Q_{n-1}$ are "the conserved quantities" of the Toda lattice, which give rise to its integrability. (For further explanation of Toda lattices we refer to [Ol-Pe], [Pe], [Re-Se].) The result of [Gi-Ki] is that the (small) quantum cohomology algebra of

$$F_n = \{E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = \mathbb{C}^n \mid E_i \text{ is an } i\text{-dimensional linear subspace of } \mathbb{C}^n\ \}$$

is

$$QH^*F_n \ \cong\ \mathbb{C}[X_1, \ldots, X_n, Q_1, \ldots, Q_{n-1}]/\langle O_n^0, O_n^1, \ldots, O_n^{n-1}\rangle,$$

where $X_1, \ldots, X_n, Q_1, \ldots, Q_{n-1}$ are regarded now as indeterminates. In other words, the conserved quantities of the open one-dimensional Toda lattice are precisely the defining relations for the quantum cohomology algebra of F_n. This remarkable fact has been explored in a number of very interesting papers (such as [Gi2], [Ki2], [Ko1], [Ko2], [Fo-Ge-Po]).

The periodic one-dimensional Toda lattice is a differential equation of the form

$$\dot{\mathcal{L}}_n(t) = [\mathcal{L}_n(t), \mathcal{M}_n(t)]$$

where $\mathcal{L}_n$ is the matrix

$$\mathcal{L}_n = \begin{pmatrix} X_1 & Q_1 & 0 & \cdots & \cdots & \cdots & -z \\ -1 & X_2 & Q_2 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & X_3 & Q_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & X_{n-2} & Q_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & -1 & X_{n-1} & Q_{n-1} \\ Q_n/z & \cdots & \cdots & \cdots & 0 & -1 & X_n \end{pmatrix}$$

and where z is a "spectral parameter" in $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Thus, $\mathcal{L}_n$ may be interpreted as a function of the real variable t with values in the loop algebra $\mathrm{Map}(S^1, M_n\mathbb{C})$. The variables $X_1, \ldots, X_n$ and $Q_1, \ldots, Q_n$ here are functions of a real variable t with $Q_i < 0$, and we assume that $X_1 + \cdots + X_n = 0$ and that $Q_1 Q_2 \cdots Q_n$ is constant. Let

$$\det(\mathcal{L}_n + \mu I) = P_n = \sum_{i=0}^{n} P_n^i \mu^i + A_n \frac{1}{z} + B_n z$$

where P_n^k, A_n, B_n are polynomials in $X_1, \ldots, X_n$ and $Q_1, \ldots, Q_n$. The $P_n^0, P_n^1, \ldots, P_n^{n-1}$ are "the conserved quantities" of the periodic Toda lattice.

The loop group $LSU_n = \mathrm{Map}(S^1, SU_n)$ plays an analogous role here to that of the group SU_n for the open Toda lattice, and the periodic flag manifold $Fl^{(n)}$ is analogous to F_n (it is a complete flag manifold for an affine Kac-Moody group). For a precise definition of $Fl^{(n)}$ we refer to section 8.7 of [Pr-Se]; we just remark that it is related to the Grassmannian model $Gr^{(n)}$ of the based loop group $\Omega SU_n \cong LSU_n/SU_n$ as follows:

$$Fl^{(n)} \;=\; \{W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n \mid W_i \in Gr^{(n)}, \mathrm{virt.dim}\, W_i = i - n, \lambda W_n = W_0\}.$$

Here, $Gr^{(n)}$ is a certain subspace of the Grassmannian of all linear subspaces of the Hilbert space

$$H = L^2(S^1, \mathbb{C}^n) = \bigoplus_{i \in \mathbb{Z}} \lambda^i \mathbb{C}^n,$$

and λW_n denotes the result of applying the linear "multiplication operator" λ (of H) to W_n. The virtual dimension is defined by $\mathrm{virt.dim}\, W = \dim(W \cap H_-) - \dim(W^\perp \cap H_+)$, where $H_+ = \oplus_{i \geq 0} \lambda^i \mathbb{C}^n$, $H_- = \oplus_{i<0} \lambda^i \mathbb{C}^n$.

Let $H^\sharp Fl^{(n)}$ denote the subalgebra of the cohomology algebra $H^* Fl^{(n)}$ which is generated by $H^2 Fl^{(n)}$. Let $QH^\sharp Fl^{(n)}$ denote the subalgebra of the quantum cohomology algebra $QH^* Fl^{(n)}$ which is generated by $H^2 Fl^{(n)}$. Then our result is:

$$QH^\sharp Fl^{(n)} \;\cong\; \mathbb{C}[Y_1, \ldots, Y_n, Q_1, \ldots, Q_n] / \langle P_n^0, P_n^1, \ldots, P_n^{n-1} \rangle,$$

where $Y_1, \ldots, Y_n$ are related to $X_1, \ldots, X_n$ by $X_i = Y_i - Y_{i-1}$ and $Y_0 = Y_n$ (the precise nature of $Y_1, \ldots, Y_n$ will be made clear later).

We refrain from calling this a "theorem", as it depends on two provisional assumptions which we shall not attempt to justify in this paper. These are (1) that a rigorous definition of $QH^\sharp Fl^{(n)}$ is possible, and (2) that $QH^\sharp Fl^{(n)}$ and $H^\sharp Fl^{(n)} \otimes \mathbb{C}[Q_1, \ldots, Q_n]$ are isomorphic as $\mathbb{C}[Q_1, \ldots, Q_n]$-modules. Regarding (1), we have little doubt that an appropriate definition can be given, for example as in [Be], [Ci1], using "quantum Schubert calculus".

Assumption (2) may be avoided, as we shall explain at the end of the paper. Our calculation is quite short, and it gives simultaneously another proof of the result of Givental and Kim for F_n (where assumptions (1) and (2) are unnecessary).

To conclude this introduction, we comment on two special features of $QH^*Fl^{(n)}$ which are not present in the case of QH^*F_n:

(i) The space $Fl^{(n)}$ — and the space of rational curves in $Fl^{(n)}$ of fixed degree — is infinite-dimensional. On the other hand, the space of rational curves of fixed degree in $Fl^{(n)}$ *which intersect a fixed finite-dimensional subvariety* is finite-dimensional. (This is an observation of [At].) It is this property which is primarily responsible for the existence of the quantum cohomology of $Fl^{(n)}$. An alternative manifestation of this property is that the first Chern class of $Fl^{(n)}$ is *finite* (see [Fr]).

(ii) Because of (i), Poincaré duality is not immediately available for $Fl^{(n)}$. However, as a substitute, we use the existence of dual Birkhoff and Bruhat cells in $Fl^{(n)}$ (see [Pr-Se]). Bruhat cells are finite-dimensional and their closures represent a basis for the homology classes of $Fl^{(n)}$; Birkhoff cells are finite-codimensional and their closures represent a basis for the the cohomology classes. These play the role of Schubert varieties and "dual" Schubert varieties in F_n.

Acknowledgements: The first author is grateful for financial support from the NSF (USA), the NCTS (National Tsing Hua University, Taiwan), and Hull University (UK). The authors thank Augustin-Liviu Mare for several helpful comments.

§1 The periodic flag manifold.

We shall review some facts concerning $Gr^{(n)}$ and $Fl^{(n)}$ from chapter 8 of [Pr-Se], and establish some additional notation. Recall that $Gr^{(n)}$ has a line bundle $\det \mathcal{W}$, which may be considered as the "top exterior power" of the tautologous bundle $\mathcal{W}$. (The fibre of $\mathcal{W}$ over $W \in Gr^{(n)}$ is W itself.) Similarly, $Fl^{(n)}$ has tautologous bundles $\mathcal{W}_i$ and associated line bundles $\det \mathcal{W}_i$.

Definition 1.1.

(1) $y_i = -c_1 \det \mathcal{W}_i \in H^2 Fl^{(n)}$

(2) $x_i = -c_1 \mathcal{W}_i/\mathcal{W}_{i-1} = y_i - y_{i-1} \in H^2 Fl^{(n)}$.

The bundles $\mathcal{W}_i/\mathcal{W}_{i-n}$ are known to be topologically trivial, so we have $y_0 = y_n$ and $x_0 = x_n$. Since $\mathcal{W}_n/\mathcal{W}_0$ is topologically equivalent to $\oplus_{i=1}^n \mathcal{W}_i/\mathcal{W}_{i-1}$, it follows that the elementary symmetric functions of $x_1, \dots, x_n$ are zero.

Let $\mathbb{C}[Y_1, \dots, Y_n]$ be the algebra of complex polynomials in certain variables $Y_1, \dots, Y_n$.

The "classical evaluation map"

$$ev_c : \mathbb{C}[Y_1, \ldots, Y_n] \to H^\sharp Fl^{(n)}, \quad Y_i \mapsto y_i$$

is by definition an epimorphism, and we shall investigate its kernel. For this we need the elementary symmetric polynomials in $X_1, \ldots, X_n$, where $X_i = Y_i - Y_{i-1}$ $(i \in \mathbb{Z}/n\mathbb{Z})$:

Definition 1.2. $S_n = \sum_{i=0}^n S_n^i \mu^i = (X_1 + \mu) \cdots (X_n + \mu)$.

We shall show that the S_n^i are generators of the kernel of ev_c, i.e. the relations defining the algebra $H^\sharp Fl^{(n)}$.

Proposition 1.3. *We have $H^\sharp Fl^{(n)} \cong \mathbb{C}[Y_1, \ldots, Y_n]/\langle S_n^0, S_n^1, \ldots, S_n^{n-1}\rangle$, the isomorphism being induced by ev_c.*

Proof. Since the bundle $\mathcal{W}_n/\mathcal{W}_0$ is topologically trivial, the map $\pi_n : Fl^{(n)} \to Gr^{(n)}$ given by $\pi(W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n) = W_n$ defines a trivial bundle over the "identity component" $Gr_{\mathrm{id}}^{(n)}$ of $Gr^{(n)}$ consisting of subspaces of virtual dimension zero. Now, $Gr_{\mathrm{id}}^{(n)}$ is homotopy equivalent to $\Omega SU_n = LSU_n/SU_n$ (see [Pr-Se]), and its cohomology is well known (see [Bo]). The fibre of the bundle is F_n. Hence $H^\sharp Fl^{(n)} \cong H^* F_n \otimes H^\sharp \Omega SU_n$, and this leads to the stated result. ■

We shall make use of the Birkhoff "cells" $\Sigma_{\mathbf{a}}$ and the Bruhat cells $C_{\mathbf{a}}$ of the Grassmannian $Gr^{(n)}$, which were introduced in section 8.4 of [Pr-Se]. They are indexed by elements $\mathbf{a}$ of $\mathbb{Z}^n$. The closures $\bar{C}_{\mathbf{a}}$ of the Bruhat cells are finite-dimensional projective algebraic varieties, and their fundamental homology classes form a system of additive generators for $H_* Gr^{(n)}$. There is a duality between the $\bar{C}_{\mathbf{a}}$'s and the $\bar{\Sigma}_{\mathbf{a}}$'s which is analogous to the duality between "opposite" Schubert decompositions of a finite-dimensional Grassmannian. This may be expressed in terms of intersections — see Theorem 8.4.5 of [Pr-Se]. The finite-codimensional varieties $\bar{\Sigma}_{\mathbf{a}}$ can be considered as representatives of a system of additive generators for $H^* Gr^{(n)}$.

For example, if $Gr_{\mathrm{id}}^{(n)}$ denotes the component of $Gr^{(n)}$ consisting of subspaces of virtual dimension zero as in the proof above, and $\mathcal{W}_{\mathrm{id}}$ denotes the restriction of the bundle $\mathcal{W}$ to $Gr_{\mathrm{id}}^{(n)}$, then the cohomology class $y = -c_1 \det \mathcal{W}_{\mathrm{id}} \in H^2 Gr_{\mathrm{id}}^{(n)}$ corresponds to the unique Birkhoff variety of codimension one in $Gr_{\mathrm{id}}^{(n)}$, in the sense that the latter is the zero set of a holomorphic section of $\det \mathcal{W}_{\mathrm{id}}^*$ (see section 7.7 of [Pr-Se]). This variety, which we shall denote by $\bar{\Sigma}$, is given explicitly by

$$\bar{\Sigma} = \{W \in Gr_{\mathrm{id}}^{(n)} \mid \dim W \cap H_- \geq 1\}.$$

The dual Bruhat variety $\bar{C}$ is given explicitly by

$$\bar{C} = \{W \in Gr^{(n)}_{\mathrm{id}} \mid \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_n \oplus \lambda H_+ \subseteq W \subseteq \mathbb{C}\lambda^{-1}e_n \oplus H_+\},$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$.

Birkhoff varieties $\bar{\Sigma}_w$ and Bruhat varieties $\bar{C}_w$ for the periodic flag manifold $Fl^{(n)}$ were defined in section 8.7 of [Pr-Se] in a similar way. They are indexed by elements w of the affine Weyl group of LSU_n. This time there are n Birkhoff varieties of codimension one, corresponding to the additive generators y_i of $H^2 Fl^{(n)}$, namely

$$\bar{\Sigma}_{(i)} = \{(W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n) \in Fl^{(n)} \mid \dim W_i \cap (H_- \oplus \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-i}) \geq 1\}.$$

The dual Bruhat varieties are

$$\bar{C}_{(i)} = \{(W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n) \in Fl^{(n)} \mid W_k = \mathbb{C}e_{n-k+1} \oplus \cdots \oplus \mathbb{C}e_n \oplus \lambda H_+ \text{ for } k \neq i\}.$$

The inclusion of $\bar{C}_{(i)} (\cong \mathbb{C}P^1)$ in $Fl^{(n)}$ defines a rational curve f_i, and the homotopy classes

$$q_i = [f_i]$$

form an additive basis of $\pi_2 Fl^{(n)} \cong H_2 Fl^{(n)} \cong \mathbb{Z}^n$. As usual in the construction of quantum cohomology, we shall in future use multiplicative notation $q^\alpha = q_1^{\alpha_1} \cdots q_n^{\alpha_n}$ for an element $\alpha = \alpha_1 q_1 + \cdots + \alpha_n q_n$ of $\pi_2 Fl^{(n)}$, i.e. instead of α we use the corresponding additive generator q^α of the group algebra of $\pi_2 Fl^{(n)}$.

It is easy to show (by considering the bundle $\pi_n : Fl^{(n)} \to Gr^{(n)}$) that a homotopy class q^α contains a rational curve only if $\alpha \geq 0$, i.e. $\alpha_i \geq 0$ for all i.

§2 Computations of quantum products.

Our computations of quantum products for $Fl^{(n)}$ are based on the existence of a Gromov-Witten invariant

$$\langle \bar{\Sigma}_{w_1} | \bar{\Sigma}_{w_2} | \bar{C}_{w_3} \rangle_{q^\alpha}.$$

This may be defined — naively — as the number of rational curves f in the homotopy class q^α such that

$$f(0) \in g_1 \bar{\Sigma}_{w_1}, \quad f(1) \in g_2 \bar{\Sigma}_{w_2}, \quad f(\infty) \in g_3 \bar{C}_{w_3},$$

where g_1, g_2, g_3 are "general" elements of the loop group LSU_n. As stated in the introduction, we shall *assume* that such invariants are well defined, and that they give rise to a commutative associative "quantum product" operation $\circ$ on $H^* Fl^{(n)} \otimes \mathbb{C}[q_1, \ldots, q_n]$, through the following standard procedure:

Definition 2.1. *For $u = [\bar{\Sigma}_{w_1}], v = [\bar{\Sigma}_{w_2}]$ in $H^*Fl^{(n)}$, let*

$$u \circ v = \sum_{\alpha \geq 0} (u \circ v)_\alpha q^\alpha,$$

where $(u \circ v)_\alpha \in H^*Fl^{(n)}$ is determined via its (Kronecker) products by

$$\langle (u \circ v)_\alpha, [\bar{C}_w] \rangle = \langle \bar{\Sigma}_{w_1} | \bar{\Sigma}_{w_2} | \bar{C}_w \rangle_{q^\alpha},$$

for all w in the affine Weyl group of LSU_n. We denote by $QH^*Fl^{(n)}$ the algebra with underlying $\mathbb{C}[q_1, \dots, q_n]$-module $H^*Fl^{(n)} \otimes \mathbb{C}[q_1, \dots, q_n]$ and product operation $\circ$.

We assume further that $\circ$ is a deformation of the cup product in cohomology in the sense that $(u \circ v)_0 = uv$, and that $\circ$ respects the grading defined in the usual way by

$$|xq^\alpha| = |x| + \langle c_1 Fl^{(n)}, q^\alpha \rangle$$

(where $x \in H^*Fl^{(n)}$). It follows from [Fr] that $\langle c_1 Fl^{(n)}, q^\alpha \rangle = 4 \sum_{i=1}^n \alpha_i$, and it is easy to check that this is the dimension of the space of basepoint preserving rational curves in the homotopy class q^α. We obtain $|(u \circ v)_\alpha| = |u| + |v| - 4 \sum_{i=1}^n \alpha_i$.

Note that we are assuming, in particular, that the ordinary cup product is given by intersections of (general translates of) Bruhat and Birkhoff varieties. We could not find a direct statement of this in the literature, but it appears to be known (see [Ca], [Gu], [Ha], [Ko-Ku]).

The following useful lemma says that, for a quantum product of the form $y_i^m \circ v$, each nonzero term $(y_i^m \circ v)_\alpha q^\alpha$ in the "quantum deformation" must be divisible by q_i.

Lemma 2.2. *Let $i \in \{1, \dots, n\}$, $m \in \mathbb{N}$. Let $v \in H^*Fl^{(n)}$. Write $y_i^m \circ v = y_i^m v + \sum_{\alpha > 0} (y_i^m \circ v)_\alpha q^\alpha$ (as above). If $(y_i^m \circ v)_\alpha \neq 0$, then $\alpha_i \geq 1$.*

Proof. The cohomology class y_i^m may be represented by a variety of the form

$$\bar{\Sigma}_{(i)}^m = g_1 \bar{\Sigma}_{(i)} \cap \dots \cap g_m \bar{\Sigma}_{(i)}$$

where $g_1, \dots, g_m$ are suitable elements of LSU_n; this is a subset of $Fl^{(n)}$ consisting of elements $W_0 \subseteq W_1 \subseteq \dots \subseteq W_n$ for which W_i (and only W_i) satisfies a certain condition. We claim that

$$\langle \bar{\Sigma}_{(i)}^m \mid \bar{\Sigma}' \mid \bar{C}' \rangle_{q^\alpha} \neq 0 \implies \alpha_i \geq 1,$$

for any Birkhoff variety $\bar{\Sigma}'$ and any Bruhat variety $\bar{C}'$.

If this assertion is false, there is a (nonzero) finite number of rational curves

$$f = (W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n) : \mathbb{C}P^1 \to Fl^{(n)}$$

in the homotopy class q^α, with $\alpha_i = 0$, such that

$$f(0) \in \bar{\Sigma}^m_{(i)}, \quad f(1) \in g_1\bar{\Sigma}', \quad f(\infty) \in g_2\bar{C}' \quad \text{(for some } g_1, g_2 \in LSU_n\text{)}.$$

Since $\alpha_i = 0$, W_i is constant. But then we obtain a continuous family of rational curves with the same properties, by pre-composing with fractional linear transformations ξ such that $\xi(0) = z \in \mathbb{C}P^1 - \{1, \infty\}$, $\xi(1) = 1$, $\xi(\infty) = \infty$. This is a contradiction. ■

We shall be interested in the "quantum versions" QS^i_n of the relations S^i_n of $H^\sharp Fl^{(n)}$. These will be the relations for the algebra $QH^\sharp Fl^{(n)}$, which is defined analogously to $H^\sharp Fl^{(n)}$ as the subalgebra of $QH^* Fl^{(n)}$ generated by $H^2 Fl^{(n)}$ (but see assumptions (1) and (2) of the introduction). By an argument of Siebert and Tian (Theorem 2.2 of [Si-Ti]), QS^i_n is obtained by suitably modifying S^i_n. To explain this modification, the following two facts are needed:

(1) Any quantum product can be expressed as a linear combination of classical products with coefficients in $\mathbb{C}[q_1, \ldots, q_n]$.

(2) Any classical product can be expressed as a linear combination of quantum products with coefficients in $\mathbb{C}[q_1, \ldots, q_n]$.

The first is obvious from the definition of quantum product, and the second may be proved by an induction argument, bearing in mind that the degree of a quantum product is the sum of the degrees of the individual factors.

Let $\mathbb{C}[Y_1, \ldots, Y_n, Q_1, \ldots, Q_n]$ be the algebra of complex polynomials in certain variables $Y_1, \ldots, Y_n, Q_1, \ldots, Q_n$. We define the quantum evaluation map

$$ev_q : \mathbb{C}[Y_1, \ldots, Y_n, Q_1, \ldots, Q_n] \to QH^\sharp Fl^{(n)}$$

as the algebra epimorphism that sends Y_i to y_i and Q_i to q_i in $QH^\sharp Fl^{(n)}$. Via the module identification $QH^\sharp Fl^{(n)} \cong H^\sharp Fl^{(n)} \otimes \mathbb{C}[q_1, \ldots, q_n]$, ev_q can be regarded as the map which evaluates all the quantum products in a "quantum polynomial" involving $y_1, \ldots, y_n, q_1, \ldots, q_n$. The classical evaluation map ev_c extends to an algebra epimorphism

$$ev_c : \mathbb{C}[Y_1, \ldots, Y_n, Q_1, \ldots, Q_n] \to H^\sharp Fl^{(n)} \otimes \mathbb{C}[q_1, \ldots, q_n].$$

In general, ev_c and ev_q do not coincide, of course. But it follows from (1) and (2) above that, for any polynomial R, there is a (not in general unique) polynomial $\mathcal{R}$ such that

$ev_c R = ev_q \mathcal{R}$. Our main computational result is that there is a simple algebraic formula for the polynomial $\mathcal{R} = QS_n^i$ in terms of the polynomial $R = S_n^i$. It will be convenient to express this in terms of the differential operators

$$\delta_i = Id - Q_i \frac{\partial^2}{\partial X_i \partial X_{i+1}}, \quad D_i = Id + Q_i \frac{\partial^2}{\partial X_i \partial X_{i+1}}, \quad i \in \mathbb{Z}/n\mathbb{Z}.$$

These operators commute (since they have constant coefficients).

Denote by V the $\mathbb{C}[Q_1, \ldots, Q_n]$-submodule of $\mathbb{C}[Y_1, \ldots, Y_n, Q_1, \ldots, Q_n]$ that is generated by elements of the form $X_{i_1} \cdots X_{i_k}$, $1 \le i_1 < \cdots < i_k \le n$.

Proposition 2.3. *On V we have*

(1) $ev_q = ev_c \delta_n \delta_{n-1} \cdots \delta_1$.

(2) $ev_c = ev_q D_n D_{n-1} \cdots D_1$.

We shall postpone the proof of Proposition 2.3 for a moment. Part (2) gives our explicit formula for $\mathcal{R}$ in terms of R, namely $\mathcal{R} = D_n D_{n-1} \cdots D_1 R$. Applying this to the relations $R = S_n^i$ we obtain the required quantum modifications:

Definition 2.4. $QS_n = \sum_{i=0}^n QS_n^i \mu^i = D_n D_{n-1} \cdots D_1 S_n$.

Corollary 2.5. *Subject to the validity of assumptions (1) and (2) of the introduction, we have $QH^\sharp Fl^{(n)} \cong \mathbb{C}[Y_1, \ldots, Y_n, Q_1, \ldots, Q_n] / \langle QS_n^0, QS_n^1, \ldots, QS_n^{n-1} \rangle$, the isomorphism being induced by ev_q.*

For example, $S_n^{n-1} = \sum_i X_i = QS_n^{n-1}$ and

$$S_n^{n-2} = \sum_{i<j} X_i X_j, \quad QS_n^{n-2} = \sum_{i<j} X_i X_j + \sum_i Q_i.$$

The relation QS_n^{n-2} corrresponds to the quantum multiplication formula $\sum_{i<j} x_i \circ x_j = \sum_{i<j} x_i x_j - \sum_i q_i$. The formula can be established by showing that $x_i \circ x_{i+1} = x_i x_{i+1} - q_i$ for all i and $x_i \circ x_j = x_i x_j$ when $j > i+1$. This, and its generalization to products of the form $x_{i_1} \circ \cdots \circ x_{i_k}$, is the basis of our proof of Proposition 2.3.

Proof of Proposition 2.3. We have $D_i \delta_i = Id$ on V, since $D_i \delta_i = Id - Q_i^2 \frac{\partial^4}{\partial X_i^2 \partial X_{i+1}^2}$, and the second term vanishes on V. If we assume (1), then we have

$$\begin{aligned} ev_q(D_n D_{n-1} \cdots D_1 X_{i_1} \cdots X_{i_k}) &= ev_c(\delta_n \delta_{n-1} \cdots \delta_1 D_n D_{n-1} \cdots D_1 X_{i_1} \cdots X_{i_k}) \\ &= ev_c(X_{i_1} \cdots X_{i_k}). \end{aligned}$$

So (2) is a consequence of (1).

To prove (1), it suffices to show that the quantum product

$$x_{i_1} \circ \cdots \circ x_{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n$$

is obtained by replacing (in any order) each occurrence of $X_i X_{i+1}$ by $X_i X_{i+1} - Q_i$ and then applying the classical evaluation map ev_c. For example, $X_1X_2X_3$ becomes $X_1X_2X_3 - X_3Q_1 - X_1Q_2 - X_2Q_3$. In terms of $y_1, \dots, y_n$ (using $x_i = y_i - y_{i-1}$) this is equivalent to:

Lemma 2.6. *Let a and b be nonnegative integers. Then*

$$(y_{i_1} \circ y_{i_1}) \circ \cdots \circ (y_{i_a} \circ y_{i_a}) \circ y_{j_1} \circ \cdots \circ y_{j_b} = (y_{i_1}^2 + q_{i_1}) \cdots (y_{i_a}^2 + q_{i_a}) y_{j_1} \cdots y_{j_b}$$

provided that all the indices in this expression are distinct and no two of $i_1, \dots, i_a$ are consecutive (mod n).

Proof. We use induction on $a+b$. For $a+b \le 1$, the only nontrivial case to be established is $y_i \circ y_i = y_i^2 + q_i$. By Lemma 2.2, each term in the quantum deformation of $y_i \circ y_i$ must contain q_i. Hence $y_i \circ y_i = y_i^2 + \lambda q_i$, where (by definition of the quantum product) we have $\lambda = \langle \bar\Sigma_{(i)} | \bar\Sigma_{(i)} | \operatorname{point} \rangle_{q_i}$. This is evaluated by counting rational curves $f = (W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n)$ with $W_k(z)$ constant if $k \ne i$. But this is essentially the Gromov-Witten invariant $\langle \operatorname{point} | \operatorname{point} | \operatorname{point} \rangle_1$ in the quantum cohomology of $\mathbb{C}P^1$, so $\lambda = 1$.

Now we proceed to the inductive step. By the previous paragraph, we have

$$(y_{i_1} \circ y_{i_1}) \circ \cdots \circ (y_{i_a} \circ y_{i_a}) \circ y_{j_1} \circ \cdots \circ y_{j_b} = (y_{i_1}^2 + q_{i_1}) \circ \cdots \circ (y_{i_a}^2 + q_{i_a}) \circ y_{j_1} \circ \cdots \circ y_{j_b}$$

It suffices to show that

$$y_{i_1}^2 \circ \cdots \circ y_{i_a}^2 \circ y_{j_1} \circ \cdots \circ y_{j_b} = y_{i_1}^2 \cdots y_{i_a}^2 y_{j_1} \cdots y_{j_b},$$

i.e. that products of the form $y_{i_1}^2 \circ \cdots \circ y_{i_a}^2 \circ y_{j_1} \circ \cdots \circ y_{j_b}$ have no "quantum deformation". (If $a = 0$ this is the same as the statement that we wish to prove; if $a > 0$ it implies the required statement, because of the induction hypothesis.)

We shall give a separate inductive argument for the last statement. For $a+b \le 1$ there is nothing to prove. For the inductive step, we consider first the case where $b > 0$. By the inductive hypothesis, we have

$$\begin{aligned} y_{i_1}^2 \circ \cdots \circ y_{i_a}^2 \circ y_{j_1} \circ \cdots \circ y_{j_b} &= y_{i_p}^2 \circ (y_{i_1}^2 \cdots \hat{y}_{i_p}^2 \cdots y_{i_a}^2 y_{j_1} \cdots y_{j_b}) \\ &= y_{j_q} \circ (y_{i_1}^2 \cdots y_{i_a}^2 y_{j_1} \cdots \hat{y}_{j_q} \cdots y_{j_b}). \end{aligned}$$

Applying Lemma 2.2, we see that each term of the quantum deformation of the left hand side must contain $q_{i_1}\cdots q_{i_a}q_{j_1}\cdots q_{j_b}$. The former has degree $4a+2b$, and the latter has degree $4a+4b$. Since $b>0$, this means that there is in fact no quantum deformation.

It remains to prove the inductive step in the case where $b=0$. By the inductive hypothesis and Lemma 2.2 again, we have

$$\begin{aligned} y_{i_1}^2\circ\cdots\circ y_{i_a}^2 &= y_{i_r}^2\circ(y_{i_1}^2\cdots\hat{y}_{i_r}^2\cdots y_{i_a}^2) \text{ for any } r\\ &= y_{i_1}^2\cdots y_{i_a}^2+\lambda q_{i_1}\cdots q_{i_a}. \end{aligned}$$

The coefficient λ here is equal to $\langle\bar{\Sigma}_{(i_r)}^2|\bar{\Sigma}'|\text{point}\rangle_{q_{i_1}\cdots q_{i_a}}$, where $\bar{\Sigma}'$ denotes the dual homology class to $y_{i_1}^2\cdots\hat{y}_{i_r}^2\cdots y_{i_a}^2$. This means we are counting rational curves $f=(W_0\subseteq W_1\subseteq\cdots\subseteq W_n)$ with $W_i(z)$ constant if $i\notin\{i_1,\ldots,i_a\}$.

Since no two of $i_1,\ldots,i_a$ are consecutive (mod n), f may be identified with a rational curve in a product $\mathbb{C}P^1\times\cdots\times\mathbb{C}P^1$ of complex projective lines. It follows that $\lambda=0$, as required. ■

§3 The periodic Toda lattice.

We are now ready to prove that the relations in the algebra $QH^\sharp Fl^{(n)}$ are equal to the conserved quantities of the periodic Toda lattice:

Theorem 3.1. *For $0\le k\le n$, we have $QS_n^k=P_n^k$.*

This will be an immediate consequence of Definition 2.4, Corollary 2.5, and part (2) of the following Proposition 3.2. We use the notation O_n,P_n from the introduction.

Proposition 3.2.

(1) $O_n=D_{n-1}D_{n-2}\cdots D_1S_n$.

(2) $P_n=D_nD_{n-1}\cdots D_1S_n+(-1)^{n+1}\frac{Q_1Q_2\cdots Q_n}{z}+z$.

Proof. (1) Expanding $O_{k+1} = \det(L_{k+1} + \mu I)$ along the last row, we have

$$O_{k+1} = (X_{k+1}+\mu)\begin{vmatrix} X_1+\mu & \cdots & \cdots & 0 \\ -1 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & X_{k-1}+\mu & Q_{k-1} \\ 0 & \cdots & -1 & X_k+\mu \end{vmatrix} + \begin{vmatrix} X_1+\mu & \cdots & \cdots & 0 \\ -1 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & X_{k-1}+\mu & 0 \\ 0 & \cdots & -1 & Q_k \end{vmatrix}$$

$$= (X_{k+1}+\mu)O_k + Q_k\left(\frac{\partial}{\partial X_k}O_k\right)$$

$$= (X_{k+1}+\mu)O_k + Q_k\frac{\partial^2}{\partial X_k \partial X_{k+1}}\{(X_{k+1}+\mu)O_k\}$$

$$= D_k\{(X_{k+1}+\mu)O_k\}.$$

Since $(X_j+\mu)D_iF = D_i\{(X_j+\mu)F\}$ for any polynomial F if $j \geq i+2$, part (1) now follows by induction.

(2) The only difference between P_n and O_n is that additional entries $-z$ and Q_n/z appear in the top right and bottom left corners of the determinant. Expanding (partially) along the last row, we see that P_n is equal to

$$\begin{vmatrix} X_1+\mu & Q_1 & \cdots & \cdots & -z \\ -1 & X_2+\mu & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & X_{n-2}+\mu & Q_{n-2} & 0 \\ 0 & \cdots & -1 & X_{n-1}+\mu & Q_{n-1} \\ 0 & \cdots & 0 & -1 & X_n+\mu \end{vmatrix} + \begin{vmatrix} X_1+\mu & Q_1 & \cdots & \cdots & -z \\ -1 & X_2+\mu & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & X_{n-2}+\mu & Q_{n-2} & 0 \\ 0 & \cdots & -1 & X_{n-1}+\mu & Q_{n-1} \\ Q_n/z & \cdots & 0 & 0 & 0 \end{vmatrix}.$$

Applying the same procedure to the right hand columns, the first determinant becomes

$$\begin{vmatrix} X_1+\mu & Q_1 & \cdots & \cdots & 0 \\ -1 & X_2+\mu & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & X_{n-2}+\mu & Q_{n-2} & 0 \\ 0 & \cdots & -1 & X_{n-1}+\mu & Q_{n-1} \\ 0 & \cdots & 0 & -1 & X_n+\mu \end{vmatrix} + \begin{vmatrix} X_1+\mu & Q_1 & \cdots & \cdots & -z \\ -1 & X_2+\mu & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & X_{n-2}+\mu & Q_{n-2} & 0 \\ 0 & \cdots & -1 & X_{n-1}+\mu & 0 \\ 0 & \cdots & 0 & -1 & 0 \end{vmatrix}$$

while the second determinant (after expansion along the last row) becomes

$$(-1)^{n+1}\frac{Q_n}{z}\begin{vmatrix} Q_1 & \cdots & \cdots & \cdots & 0 \\ X_2+\mu & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & X_{n-2}+\mu & Q_{n-2} & 0 \\ 0 & \cdots & -1 & X_{n-1}+\mu & Q_{n-1} \end{vmatrix} + (-1)^{n+1}\frac{Q_n}{z}\begin{vmatrix} Q_1 & \cdots & \cdots & \cdots & -z \\ X_2+\mu & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & X_{n-2}+\mu & Q_{n-2} & 0 \\ 0 & \cdots & -1 & X_{n-1}+\mu & 0 \end{vmatrix}.$$

The last term here is

$$(-1)^{n+1}\frac{Q_n}{z}(-1)^n(-z)\begin{vmatrix} X_2+\mu & \cdots & \cdots & \cdots \\ \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & X_{n-2}+\mu & Q_{n-2} \\ \cdots & \cdots & -1 & X_{n-1}+\mu \end{vmatrix} = Q_n \frac{\partial^2}{\partial X_n \partial X_1} O_n.$$

Taking the sum of all four terms, we have

$$\begin{aligned} P_n &= O_n + z + (-1)^{n+1}\frac{Q_n}{z} Q_1 \cdots Q_{n-1} + Q_n \frac{\partial^2}{\partial X_n \partial X_1} O_n \\ &= D_n O_n + z + (-1)^{n+1}\frac{Q_1 \cdots Q_n}{z}. \end{aligned}$$

The required formula for P_n follows from this and (1). ■

§4 Remarks.

Our computation of quantum products in §2 recovers the result of Givental and Kim for F_n. To see this, we denote by $I : F_n \to Fl^{(n)}$ the inclusion of the fibre

$$\pi_n^{-1}(H_+) = \{ \ \lambda H_+ \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq H_+ \in Fl^{(n)} \ \} \cong F_n.$$

Let $\hat{y}_i = I^* y_i$ and $\hat{x}_i = I^* x_i$, where $1 \le i \le n$ as usual. Observe that $\hat{y}_0 = \hat{y}_n = 0$ now, and so $\hat{y}_i = \hat{x}_1 + \cdots + \hat{x}_i$ when $1 \le i \le n$.

In this situation we have evaluation maps

$$\begin{aligned} &\hat{ev}_q : \mathbb{C}[Y_1, \dots, Y_{n-1}, Q_1, \dots, Q_{n-1}] \to QH^* F_n \\ &\hat{ev}_c : \mathbb{C}[Y_1, \dots, Y_{n-1}, Q_1, \dots, Q_{n-1}] \to H^* F_n \otimes \mathbb{C}[q_1, \dots, q_{n-1}], \end{aligned}$$

given by $Y_i \mapsto \hat{y}_i$ and $Q_i \mapsto q_i$, $1 \le i \le n-1$. We denote by $\hat{V}$ the $\mathbb{C}[Q_1, \dots, Q_{n-1}]$-submodule of $\mathbb{C}[Y_1, \dots, Y_{n-1}, Q_1, \dots, Q_{n-1}]$ that is generated by elements of the form $X_{i_1} \cdots X_{i_k}$, $1 \le i_1 < \cdots < i_k \le n$ where $X_i = Y_i - Y_{i-1}$ (and $Y_0 = Y_n = 0$). The analogue of Proposition 2.3 is then:

Proposition 4.1. *On $\hat{V}$ we have*

(1) $\hat{ev}_q = \hat{ev}_c \delta_{n-1} \cdots \delta_1$.

(2) $\hat{ev}_c = \hat{ev}_q D_{n-1} \cdots D_1$.

Proof. The difference between the current situation and the situation of Proposition 2.3 is that the expression $\hat{y}_n \circ \hat{y}_n$ contributes nothing to the quantum deformation. Since the

only other appearances of $\hat{y}_n$ are linear, the result is the same as for Proposition 2.3 but with the final δ_n (and D_n) omitted. ■

Let $\hat{QS}_n = \sum_{i=0}^n \hat{QS}_n^i \mu^i = D_{n-1} \cdots D_1 S_n$. By the argument of [Si-Ti], the coefficients $\hat{QS}_n^i$ are the defining relations for $QH^* F_n$.

Corollary 4.2 (Givental and Kim). *For $0 \le k \le n$, we have $\hat{QS}_n^k = O_n^k$.*

Proof. Proposition 4.1 and part (1) of Proposition 3.2. ■

A comment is necessary on our "differential operator formulae" for the quantum relations in the case of F_n (Proposition 4.1). When we discovered these formulae we believed (naively) that they were new. However, after completing our calculations, we became aware of (i) the papers [Sa-Ko], [Wo] in which similar formulae were given for the conserved quantities of the Toda lattice, and (ii) the paper [Ci2] (containing full details of the results announced in [Ci1]) in which similar formulae were obtained for the quantum relations of F_n as a consequence of a general theory of quantum Schubert calculus.

Returning to the infinite-dimensional case, it should be said that the relation between $QH^\sharp Fl^{(n)}$ and the periodic Toda lattice is a plausible extension of the formula of Givental and Kim, in view of the following two facts:

(a) Formally, the open Toda lattice may be obtained from the periodic Toda lattice by setting $Q_n = 0$.

(b) For a finite-dimensional fibre bundle, formula (2.17) of [As- Sa] says that the quantum cohomology of the fibre should be obtained by dividing the vertical quantum cohomology of the total space by the cohomology (in positive dimensions) of the base. Applying this to the bundle $\pi_n : Fl^{(n)} \to Gr^{(n)}$ amounts to setting $q_n = 0$ (to obtain the vertical quantum cohomology) and then $y_n = 0$. Our formula is consistent with this procedure.

In this paper we have focused attention on $QH^\sharp Fl^{(n)}$, "the subalgebra of $QH^* Fl^{(n)}$ generated by two-dimensional classes", in accordance with the philosophy of [Gi-Ki], [Au]. However, as we remarked in the introduction, it is not *a priori* clear whether $QH^\sharp Fl^{(n)}$ is the same as $H^\sharp Fl^{(n)} \otimes \mathbb{C}[q_1, \dots, q_n]$, i.e. whether the latter space is closed under quantum multiplication. The smaller subspace $H^* F_n \otimes \mathbb{C}[q_1, \dots, q_n]$ is in fact more appropriate from the point of view of the periodic Toda lattice, and its use will render assumption (2) of the introduction unnecessary. (Note that $H^* F_n$ is a subalgebra of the ordinary cohomology algebra $H^* Fl^{(n)}$ because $Fl^{(n)}$ is diffeomorphic to $F_n \times Gr^{(n)}$.) It may be shown by the methods of this paper that $H^* F_n \otimes \mathbb{C}[q_1, \dots, q_n]$ is closed under our hypothetical quantum product, and hence that we have an isomorphism of algebras

$$H^* F_n \otimes \mathbb{C}[q_1, \dots, q_n] \cong \mathbb{C}[X_1, \dots, X_n, Q_1, \dots, Q_n] / \langle QS_n^0, QS_n^1, \dots, QS_n^{n-1} \rangle.$$

Furthermore, this algebra is the "coordinate ring" of a spectral cover (in the sense of [Au]), which is in turn exactly the zero level set of the conserved quantities of the periodic Toda lattice.

Finally, we point out that the quantum cohomology calculation of Lemma 2.6 — the main ingredient of Propositions 2.3 and 4.1 — amounts to an inductive procedure whereby certain quantum products in a full flag manifold are reduced to quantum products in products of flag manifolds of lower rank. This seems likely to work quite generally for flag manifolds of the form G/B or LG/B.

References

[As-Sa] A. Astashkevich and V. Sadov, *Quantum cohomology of partial flag manifolds*, Comm. Math. Phys. **170** (1995), 503–528.

[At] M.F. Atiyah, *Instantons in two and four dimensions*, Comm. Math. Phys. **93** (1984), 437–451.

[Au] M. Audin, *Symplectic geometry in Frobenius manifolds and quantum cohomology*, J. Geom. and Physics **25** (1998), 183–204.

[Be] A. Bertram, *Quantum Schubert calculus*, Advances in Math. **128** (1997), 289–305.

[Bo] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35–61.

[Ca] J.B. Carrell, *Vector fields, flag varieties, and Schubert calculus*, Proceedings of the Hyderabad Conference on Algebraic Groups (S. Ramanan, ed.), Manoj Prakashan, Madras, 1991, pp. 23–57.

[Ci1] I. Ciocan-Fontanine, *Quantum cohomology of flag varieties*, Internat. Math. Res. Notices **6** (1995), 263–277.

[Ci2] I. Ciocan-Fontanine, *The quantum cohomology ring of flag varieties*, Trans. Amer. Math. Soc. **351** (1999), 2695–2729.

[Fo-Ge-Po] S. Fomin, S. Gelfand, and A. Postnikov, *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10** (1997), 565–596.

[Fr] D.S. Freed, *The geometry of loop groups*, J. Differential Geom. **28** (1988), 223–276.

[Gi1] A.B. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices **13** (1996), 613–663.

[Gi2] A. Givental, *Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture*, Topics in Singularity Theory (A. Khovanskii, A. Varchenko, and V. Vassiliev, eds.), AMS Translations 180, Amer. Math. Soc., 1997, pp. 103–115.

[Gi-Ki] A. Givental and B. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, Comm. Math. Phys. **168** (1995), 609–641.

[Gu] E. Gutkin, *Schubert calculus on flag varieties of Kac-Moody groups*, Algebras, Groups and Geometries **3** (1986), 27-59.

[Ha] Z. Haddad, *A Coxeter group approach to Schubert varieties*, Infinite-dimensional groups with applications (V. Kac, ed.), Math. Sci. Res. Inst. Publ. 4, Springer, 1985, pp. 157–165.

[Ki1] B. Kim, *On equivariant quantum cohomology*, Internat. Math. Res. Notices **17** (1996), 841–851.

[Ki2] B. Kim, *Quantum cohomology of flag manifolds G/B and quantum Toda lattices*, Ann. of Math. **149** (1999), 129–148.

[Ko1] B. Kostant, *Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight ρ*, Selecta Math. (N.S.) **2** (1996), 43–91.

[Ko2] B. Kostant, *Quantum cohomology of the flag manifold as an algebra of rational functions on*

a unipotent algebraic group, Deformation theory and symplectic geometry (Ascona, 1996) (D. Sternheimer, J. Rawnsley and S. Gutt, eds.), Mathematical Physics Studies, 20, Kluwer, 1997, pp. 157–175.

[Ko-Ku] B. Kostant and S. Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G*, Advances in Math. **62** (1986), 187–237.

[Lu] P. Lu, *A rigorous definition of fiberwise quantum cohomology and equivariant quantum cohomology*, Comm. Anal. Geom. **6** (1998), 511–588.

[Ol-Pe] M.A. Olshanetsky and A.M. Perelomov, *Integrable systems and finite-dimensional Lie algebras*, Dynamical Systems VII (V.I. Arnol'd and S.P. Novikov, eds.), Encylopaedia of Mathematical Sciences 16, Springer, 1994, pp. 87–116.

[Pe] A.M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Birkhäuser, 1990.

[Pr-Se] A.N. Pressley and G.B. Segal, *Loop Groups*, Oxford Univ. Press, 1986.

[Re-Se] A.G. Reyman and M.A. Semenov-Tian-Shansky, *Group-theoretical methods in the theory of finite-dimensional integrable systems*, Dynamical Systems VII (V.I. Arnol'd and S.P. Novikov, eds.), Encylopaedia of Mathematical Sciences 16, Springer, 1994, pp. 116–225.

[Sa-Ko] K. Sawada and T. Kotera, *Integrability and a solution for the one-dimensional N-particle system with inversely quadratic pair potentials*, J. Phys. Soc. Japan **39** (1975), 1614–1618.

[Si-Ti] B. Siebert and G. Tian, *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, alg-geom/9403010, Asian J. Math. **1** (1997), 679–695.

[Wo] S. Wojciechowski, *Involutive set of integrals for completely integrable many-body problems with pair interaction*, Lett. Nuovo Cimento **18** (1977), 103–107.

MG:

Department of Mathematics
Graduate School of Science
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi
Tokyo 192-0397, Japan

martin@comp.metro-u.ac.jp

TO:

Department of Mathematics
Graduate School of Science and Engineering
Tokyo Institute of Technology
Okayama 2-12-1, Meguro-ku
Tokyo 152-8551, Japan

otofuji@math.titech.ac.jp